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Nonlinear equation for anomalous diffusion: Unified power-law and stretched exponential exact solution

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The nonlinear diffusion equation $\partial\rho/\partial t = D\tilde{\Delta}\rho^{\nu}$ is analyzed here, where $\tilde{\Delta} \equiv (1/r^{d-1})(\partial/\partial r) r^{d-1-\theta}\partial/\partial r$, and d, θ, and ν are real parameters. This equation unifies the anomalous diffusion equation on fractals ($\nu=1$) and the spherical anomalous diffusion for porous media ($\theta=0$). An exact point-source solution is obtained, enabling us to describe a large class of subdiffusion [$\theta>(1-\nu)d$], ''normal'' diffusion [$\theta=(1-\nu)d$] and superdiffusion [$\theta<(1-\nu)d$]. Furthermore, a thermostatistical basis for this solution is given from the maximum entropic principle applied to the Tsallis entropy.

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One of the most ubiquitous processes in nature is the diffusive one. In this context, anomalous diffusion has awakened great interest nowadays, in particular in a variety of physical applications. A representative set of such applications of current interest is surface growth and transport of fluid in porous media [1], diffusion in plasmas [2], diffusion on fractals [3], subrecoil laser cooling [4], CTAB micelles dissolved in salted water [5], two dimensional rotating flow [6], and anomalous diffusion at liquid surfaces [7]. The anomalous diffusive process is commonly characterized from the mean-square displacement time dependence, $\langle r^2\rangle \propto t^{\sigma}$, with $\sigma\neq 1$, i.e., we have superdiffusion for $\sigma>1$ and subdiffusion for $\sigma<1$. For a system that presents anomalous spreading, it is generally associated with a non-Gaussian space-time distribution, such as power-law or stretched exponential. In this framework, it is desirable to incorporate, in a unified way, these two behaviors, since it enables us to describe a wide class of diffusive processes. The present work is dedicated to giving such unified description.

Power-law or stretched exponential distributions arise naturally from generalizations of the d-dimensional diffusion equation

$$\frac{\partial\rho}{\partial t} = D\Delta\rho, \tag{1}$$

with $\rho=\rho(\overline{x},t)$, $\overline{x}=(x_1,x_2,\ldots,x_d)$, $\Delta=\Sigma_{n=1}^{d}\partial^2/\partial x_n^2$, and D being the diffusion coefficient. The nonlinear equation

$$\frac{\partial\rho}{\partial t} = D\Delta\rho^{\nu} \tag{2}$$

is just one of these generalizations, where ν is a real parameter. Equation (2) has been employed to model diffusion in porous medium (see Ref. [1] and references therein) and in connection with generalized Tsallis statistics [8–10]. Another important kind of anomalous diffusion, in a tridimensional space, is related to turbulent diffusion in the atmosphere and is usually described by [11]

$$\frac{\partial\rho}{\partial t} = \nabla(K\nabla\rho), \tag{3}$$

where $K\propto r^{4/3}$($r=|\overline{x}|$). In a more general case, we consider Eq. (3) in a d-dimensional space with $K\propto r^{-\theta}$, where θ is a real parameter. Thus, $\nabla(K\nabla)$ is proportional to $r^{-(d-1)}(\partial/\partial r)r^{d-1-\theta}\partial/\partial r + A/r^{2-\theta}$ (A is an operator depending on the angular variables), and consequently $\tilde{\Delta} \equiv r^{-(d-1)}(\partial/\partial r)r^{d-1-\theta}\partial/\partial r$ is the radial part to be considered in the study of the spherical symmetrical solutions of Eq. (3). In this context, when d is interpreted as fractal dimension in an embedding N-dimensional space, the equation

$$\frac{\partial\rho}{\partial t} = D\tilde{\Delta}\rho \tag{4}$$

has been used to study diffusion on fractals [12].

Here we are going to propose the equation

$$\frac{\partial\rho}{\partial t} = \nabla(K\nabla\rho^{\nu}), \tag{5}$$

as a unification of Eqs. (2) and (3). In fact, Eq. (5) reduces to the correlated anomalous diffusion (2) if $K=D$, and to the generalized Richardson equation (3) if $\nu=1$. The present study is mainly addressed to the point-source solution of Eq. (5), because it contains, as particular cases, a(n) (asymptotic) power-law and a stretched exponential. In this way, we focus our attention on the radial equation

$$\frac{\partial\rho}{\partial t} = D\tilde{\Delta}\rho^{\nu}. \tag{6}$$

Using this equation instead of Eq. (5) enables us to analyze cases with noninteger d, so we can relate d with a fractal dimension. Therefore, in the following discussion, we are going to consider d as a non-negative real parameter.

In order to motivate the ansatz to obtain an exact time-dependent solution for Eq. (6), we recall the corresponding solutions for equations Eqs. (1), (2), and (4). The time-dependent point-source solution for Eq. (1) is

$$\rho(r,t)=\frac{\rho_0}{(4\pi Dt)^{d/2}}\exp\left(-\frac{r^2}{4Dt}\right), \tag{7}$$

where the normalization $\Omega_d\int_0^\infty\rho(r,t)r^{d-1}dr=\rho_0$ and the d-dimensional solide angle $\Omega_d\equiv 2\pi^{d/2}/\Gamma(d/2)$ have been used. From Eq. (7) we can easily obtain the Einstein formula for the Brownian montion, i.e., $\langle r^2\rangle=2dDt$.

The analogous solution for Eq. (2) is [8–10]

$$\rho(r,t)=[1-(1-q)\beta_1(t)r^2]^{1/1-q}/Z_1(t), \tag{8}$$

where $q=2-\nu$, $Z_1(t)\propto t^{d/[2+d(1-q)]}$ and $\beta_1(t)\propto t^{-2/[2+d(1-q)]}$. It is important to emphasize the short or long tailed shape of Eq. (8), when compared with the normal diffusion (limit $q\to 1$). When $q<1$ we have $\rho(r,t)=0$ for $1-(1-q)\beta_1(t)r^2<0$, giving the short tailed behavior for $\rho(r,t)$. On the other hand, when $q>1$, the asymptotic power-law behavior for solution (8), $r^{-2/(q-1)}$, shows that $\rho(r,t)$ is a long tailed function. This short or long tailed behavior for $\rho(r,t)$ reflects directly on the mean-square displacement, leading to $\langle r^2\rangle\propto t^{2/[2+d(1-q)]}$. Again compared with the usual diffusion, $q=1$, we have a superdiffusion (subdiffusion) for $q>1(q<1)$.

The fundamental solution of Eq. (4) is the stretched exponential [12]

$$\rho(r,t)=\exp[-\beta_2(t)r^{\theta+2}]/Z_2(t), \tag{9}$$

which $Z_2(t)\propto t^{d/(\theta+2)}$ and $\beta_2(t)\propto t^{-1}$, presenting a short (long) tailed behavior for $\theta>0(\theta<0)$. Furthermore, the mean-square displacement behavior is $\langle r^2\rangle\propto t^{2/(\theta+2)}$. Thus, for $\theta>0$ ($\theta<0$) we have a subdiffusive (superdifussive) regime.

Note that Eqs. (7)–(9) can be interpolated if we employ a generalized stretched Gaussian function, i.e., $G_{(q,\lambda)}(x)\equiv[1-(1-q)|x|^\lambda]^{1/(1-q)}$ or $G_{(q,\lambda)}(x)\equiv 0$ when $[1-(1-q)|x|^\lambda]<0$, with q and λ being real parameters. In this direction, our ansatz to solve Eq. (6) is

$$\rho(r,t)=[1-(1-q)\beta(t)r^\lambda]^{1/1-q}/Z(t) \tag{10}$$

or $\rho(r,t)=0$ if $1-(1-q)\beta(t)r^\lambda<0$, where $\beta(t)$ and $Z(t)$ are functions to be determined. By using this ansatz in Eq. (6) we verify that $\beta(t)$ and $Z(t)$, with $\lambda=\theta+2$ and $q=2-\nu$, obey the equations

$$\begin{aligned}\frac{dZ(t)}{dt}&=D\lambda(2-q)d\beta(t)Z^q(t),\\ \frac{d\beta(t)}{dt}&=-D\lambda^2(2-q)\beta^2(t)Z^{q-1}(t).\end{aligned} \tag{11}$$

The solutions of these nonlinear differential equations, which lead Eqs. (7)–(9) as limit cases of Eq. (10), are

$$\beta(t)=\mathcal{A}t^{-\lambda/[\lambda+d(1-q)]},\quad Z(t)=\mathcal{B}t^{d/[\lambda+d(1-q)]}, \tag{12}$$

where

$$\begin{aligned}\mathcal{A}&=\{\gamma^{q-1}[D\lambda(2-q)(\lambda+d(1-q))]\}^{-\lambda/[\lambda+d(1-q)]},\\ \mathcal{B}&=\{\gamma[D\lambda(2-q)(\lambda+d(1-q))]^{d/\lambda}\}^{\lambda/[\lambda+d(1-q)]},\end{aligned} \tag{13}$$

which

$$\gamma=\frac{2\pi^{d/2}}{\lambda\rho_0}\frac{\Gamma\left(\frac{d}{\lambda}\right)}{\Gamma\left(\frac{d}{2}\right)}\times\begin{cases}\dfrac{\Gamma\left(\frac{1}{q-1}-\frac{d}{\lambda}\right)}{(q-1)^{d/\lambda}\Gamma\left(\frac{1}{q-1}\right)} & (q>1)\\[2ex] \dfrac{\Gamma\left(\frac{1}{1-q}+1\right)}{(1-q)^{d/\lambda}\Gamma\left(\frac{d}{\lambda}+\frac{1}{1-q}+1\right)} & (q<1).\end{cases} \tag{14}$$

The normalization condition, $\Omega_d\int_0^\infty\rho(r,t)r^{d-1}dr=\rho_0$, employed in the above calculation can only be satisfied if $\lambda>0$ and $\lambda+d(1-q)>0$. From these conditions over the parameters q and λ, we verify that the exponents in $\beta(t)$ and $Z(t)$ are respectively negative and positive. In addition to the normalization condition, the restriction $q<2$ is necessary for $\rho(r,t)$ to be real. In the following, we assume that the parameters obey the above restrictions. Of course, by setting the appropriate limits of parameters θ and ν, or equivalently λ and q, the solutions (7)–(9) are recovered, giving the full expression for $\beta_1(t)$, $Z_1(t)$, $\beta_2(t)$, and $Z_2(t)$.

By using the above solution we can calculate the mean value of r^α; it is

$$\langle r^\alpha\rangle=\frac{\int_0^\infty r^\alpha\rho(r,t)r^{d-1}dr}{\int_0^\infty\rho(r,t)r^{d-1}dr}=\mathcal{C}_\alpha t^{\alpha/[\lambda+d(1-q)]}, \tag{15}$$

where

$$\mathcal{C}_\alpha=\mathcal{A}^{-(\alpha/\lambda)}\frac{\Gamma\left(\frac{d+\alpha}{\lambda}\right)}{\Gamma\left(\frac{d}{\lambda}\right)}\times\begin{cases}\dfrac{\Gamma\left(\frac{1}{q-1}-\frac{d+\alpha}{\lambda}\right)}{(q-1)^{\alpha/\lambda}\Gamma\left(\frac{1}{q-1}-\frac{d}{\lambda}\right)} & (q>1)\\[2ex] \dfrac{\Gamma\left(\frac{d}{\lambda}+\frac{1}{1-q}+1\right)}{(1-q)^{(\alpha/\lambda)}\Gamma\left(\frac{d+\alpha}{\lambda}+\frac{1}{1-q}+1\right)} & (q<1),\end{cases} \tag{16}$$

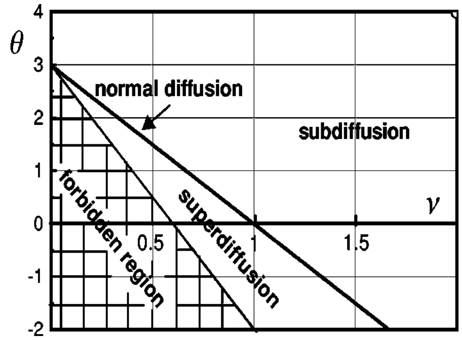

FIG. 1. Diffusive regime related to Eq. (6) in terms of its dimensionless parameters θ and ν to $d=3$. Thus, by using $\langle r^2\rangle \propto t^{2/[\theta+2+d(\nu-1)]}$ from Eq. (15) we have classified the subdiffusive $[\theta>(1-\nu)d]$, ''normal'' $[\theta=(1-\nu)d]$, and superdiffusive $[\theta<(1-\nu)d]$ regimes. The forbidden region refers to the region of parameters where $\langle r^2\rangle$ does not exist (diverges).

with $\mathcal{A}$ given by Eq. (13). When $q<1$, the mean value $\langle r^\alpha\rangle$ always exists. On the other hand, the existence of $\langle r^\alpha\rangle$ for $q>1$ imposes a further restriction over the parameters: $\lambda+d(1-q)>\alpha(q-1)$.

To decide if the diffusion is anomalous or ''normal,'' we consider Eq. (15) with $\alpha=2$. In this way, we have $\langle r^2\rangle\propto t^\sigma$ with $\sigma=2/[2+\theta+d(\nu-1)]$. Thus, the condition for ''normal'' diffusion, $\sigma=1$, can be satisfied even when ρ does not obey Eq. (1), i.e., $\theta=d(1-\nu)$ with $\theta\neq 0$ and $\nu\neq 1$. In this case, we can also verify that the anomalous diffusive regime induced by $\theta\neq 0$ is compensated by a convenient one with $\nu\neq 1$. Furthermore, this competition between θ and ν values can lead to a subdiffusion ($\sigma<1$) if $\theta>d(1-\nu)$ or a superdiffusion ($\sigma>1$) if $\theta<d(1-\nu)$. This classification is illustrated in Fig. 1 for $d=3$.

In the following, we discuss the consequences of the above classification on the ρ shape. From solution (8), for porous medium, and solution (9), for diffusion on fractals, we can see that the superdiffusive (subdiffusive) regime is associated with the long (short) tail of $\rho(r,t)$ when compared with Gaussian (7). However, this connection is not valid in general. To illustrate the relation between regime of diffusion and tail behavior of $\rho(r,t)$, we consider Fig. 2. In this figure, we plot $\rho(r,t)$ given by Eq. (10) versus r to some values of θ and ν, subject to the restriction $\theta=d(1-\nu)$ (the ''normal'' diffusion line indicated in Fig. 1). In this case we

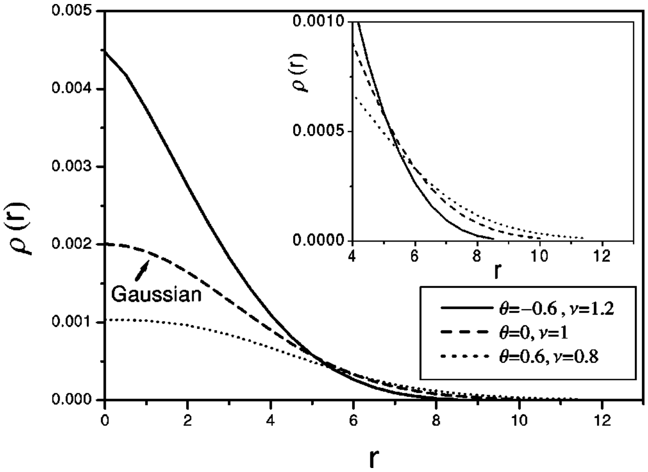

FIG. 2. By considering the ''normal'' diffusion, $\theta=(1-\nu)d$, the shape of $\rho(r,t)$ with $d=3$, $D=1$, $\rho_0=1$, and $t=5$ is illustrated in three cases: short tail ($\theta=-0.6$ and $\nu=1.2$), Gaussian ($\theta=0$ and $\nu=1$), and long tail ($\theta=0.6$ and $\nu=0.8$). Inset: detail of the tail behavior in the three cases above. The constant of normalization ρ_0 is considered dimensionless; thus, ρ is given in units of r^{-d} and D in units of $r^{\theta+2-d(1-\nu)}t^{-1}$, with r and t being measured in units of distance and time, respectively.

observed short and long tail behaviors compared with the Gaussian one.

To conclude our discussion about Eq. (5) and its radial time-dependent solution (10), we present an entropic basis for this solution. This basis is motivated by the Tsallis generalized statistical mechanics [13–15], where the Tsallis entropy [13] $S_q=(1-\Sigma_{j=1}^W p_j^q)/(q-1)$ plays a central role, with $\{p_j\}$ being the probabilities for the W states of the system, and $q\in\mathcal{R}$ being the Tsallis index (by taking the limit $q\to 1$, we recover the usual entropy $S_1=-\Sigma_{j=1}^W p_j \ln p_j$). To understand the entropic basis, for simplicity, let us consider the maximization of $S_q=[1-\int_{-\infty}^{\infty}\rho(x)^q dx]/(q-1)$ subject to the constraints [13,14] $\int_{-\infty}^{\infty}\rho(x)dx=1$ and $\int_{-\infty}^{\infty}|x|^\lambda\rho(x)dx=U_q$. This maximization leads to $\rho(x)=[1-(q-1)\beta|x|^\lambda]^{1/(q-1)}/\mathcal{Z}_q$, where $\mathcal{Z}_q=\int_{-\infty}^{\infty}[1-(q-1)\beta|x|^\lambda]^{1/(q-1)}dx$ and β is related to the Lagrange multipliers.

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